**HABILITATION THESIS**

# CONTRIBUTIONS TO THE THEORY OF ITERATED FUNCTION SYSTEMS

**Radu MICULESCU**

**Specialization: Mathematics**

**University of Bucharest**

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**Abstract**

In this thesis we present our work on iterated function systems.

The fractal theory, by its power to model objects having a complicated structure, is an extension of the classic geometry. Some exotic objects, like the triadic Cantor set, Koch curve and graph of Weierstrass function, which are on the other side of ordinary imagination, produced a huge impact in the mathematical community. B. Mandelbrot noted some common features of these mathematical object and observed that certain phenomenon from real world can be described using them, founding in this way the fractal theory. A new era of this theory began once an image of a fractal was obtained by means of a computer. Immediately engineers and physics, economic, biology researchers got interested on this topic.

One of the most common and popular framework for self-similar fractals is the theory of iterated function systems which was initiated by J. Hutchinson (see also the work of P. Moran) in his seminal paper entitled *Fractals and self similarity* published in Indiana University Mathematics Journal, in 1981, where he discussed the notion of self-similarity. A set $A$ is self-similar if it is made up of a finite transformed copies of itself. More precisely, given a complete metric space $(X, d)$ and a finite family of contractions $f_1, f_2, ..., f_n : X \to X$, the unique non-empty compact subset $A$ of $X$ having the property that $A = \bigcup_{i=1}^{n} f_i(A)$ is called a self-similar fractal (or the attractor of the iterated function system $((X, d), \{f_i\}_{i\in\{1,2,...,n\}})$). This concept, which was popularized by Michael Barnsley, especially by his famous book *Fractals everywhere*, received much attention in recent years in connection with the study of fractals.

Since it is possible to approximate any compact subset in the space $X$ by an attractor of some iterated function system, it is natural to ask the following question: which compact sets can be realized as attractors of iterated function systems. An example repertory starts with simple sets such as an interval, a square, the closure of the unit disc and continues with more exotic sets such as the Cantor ternary set, the Sierpinski gasket, the Menger sponge, the Black Spleenwort fern, the Barnsley fern, the Castle fractal, the Julia sets of quadratic transformations, the Koch curve, the Polya's curve, the Levy's curve or the Takagi graph. At the same time, it is a natural question to ask whether it is true that any compact set is actually the invariant set of some iterated function system. The answer is no. For example, L. Stacho and L. Szabo constructed compact sets in $\mathbb{R}$ that are not invariant sets for any iterated function system.

One can find self similar sets almost everywhere in the universe: galaxies, clouds, coastlines and borderlines, landscapes, human anatomy, chemical reactions, bacteria cultures, plants, data compression, market price fluctuation. Because of the variety of the applications of iterated function systems in the last years, there is a current effort to extend Hutchinson's classical framework for fractals to more general contractions and to possibly infinite iterated function systems.

After I obtained the Ph. D. title in 1999, my scientific interest was focus on iterated function systems. The purpose of this thesis is the presentation of the results that I have obtained on this topic. They can be grouped in two main thematic parts, namely *the contributions to the theory of finite iterated function systems* (contained in the first three chapters) and *the contributions to the theory of infinite iterated function systems* (contained in the next two chapters), which are detailed in the sequel. Let us mention that another way to generate fractal sets is due to Gaston Julia and Pierre Fatou who studied the iterations of rational functions on the Riemann sphere. It is not our goal to discuss this topic in the present thesis.

**Chapter I** is dedicated to the contributions of the author of this thesis to the classical theory of (finite) iterated function systems.

In *Section I.2* we recall basic definitions and well-known facts from the classical theory of iterated function systems (for short IFS).

In *Section I.3* we establish a connection between the theory of compact operators and the theory of iterated function systems. First we present a family of IFSs whose attractors are not connected and then, in contrast to the known characterizations of the compact operators which are confined to the framework of the functional analysis, we present such a characterization by means of the non-connectedness of the attractors of a family of IFSs related to the considered operators.

In *Section I.4* we present an approximation result concerning fractals generated by an iterated function system in the infinite dimensional space of continuous functions on a compact interval. More precisely, we approximate the fractal via a finite approximant set and project this approximant set in two dimensions, in order to "draw" a picture of it.

**Chapter II** is devoted to the contributions of the author of this thesis to the study of generalization of finite iterated function systems.

Given a metric space $(X, d)$, the idea of our generalization of the notion of an IFS is to consider contractions from $X^m = \mathop{\times}\limits_{k=1}^{m} X$ to $X$, rather than contractions from $X$ to itself. We call such an object a generalized iterated function system (for short GIFS), or, more accurately, a generalized iterated function system on $X$ of order $m$ (for short G$m$IFS).

In *Section II.2* we recall basic definitions and well-known facts concerning generalized contractions. We also introduce the concept of GIFS, which is a finite family of Edelstein contractions $f_k : X^m \to X$, where $(X,d)$ is a metric space and $m,n \in \mathbb{N}$, $k \in \{1,2,...,n\}$.

In *Section II.3*, using some fixed points theorem for Edelstein contractions from $X^m$ to $X$, we prove, in case that $(X,d)$ is a compact metric space and the functions $f_k$ are Edelstein contractions, the existence and the uniqueness of the attractor of such a GIFS.

*Section II.4* treats the case that $(X,d)$ is a complete metric space and the functions $f_k$ are Lipschitz contractions. We prove, in this framework, the existence and the uniqueness of the attractor of such a GIFS and explore its properties (among them we give an upper bound for the Hausdorff-Pompeiu distance between the attractors of two such GIFSs and an upper bound for the Hausdorff-Pompeiu distance between the attractors of such a GIFS and an arbitrary compact subset of $X$). Finally we present an example showing that the notion of GIFS is a natural generalization of the notion of IFS.

In **Chapter III** we present the contributions of the author of this thesis to the generalization of the notion of Hutchinson measure.

In *Section III.2* we recall basic definitions and well-known facts concerning Lipschitz functions, generalized contractions, generalized iterated function systems, the Monge-Kantorovich metric and iterated function systems with probabilities.

The Hutchinson measure is the invariant measure associated with an iterated function system with probabilities. Given a metric space $(X,d)$, generalized iterated function systems on $X$ of order 2 (G2IFS) are generalizations of iterated function systems which are obtained by considering contractions from $X \times X$ to $X$, rather than contractions from a metric space $X$ to itself. Along the lines of this generalization we consider generalized iterated function systems with probabilities (GIFSp) and generalized iterated function systems with place dependent probabilities (GIFSpdp).

In *Section III.3* we prove the existence and the uniqueness of an analogue of Hutchinson measure associated to a GIFSp. Moreover we show that the support of such a measure is the attractor of the GIFSp and we construct a sequence of measures converging, in the Monge-Kantorovich metric, to this measure.

In *Section III.4* we provide sufficient conditions under which the Markov operator associated to a GIFSpdp is Lipschitz. We also prove, under certain conditions, the existence and the uniqueness of an analogue of Hutchinson measure associated to a GIFSpdp. We show that the support of such a measure is the attractor of the generalized iterated function system, we construct a sequence of measures converging, in the Monge-Kantorovich metric, to this measure and we give an estimation of the convergence's speed of the sequence.

**Chapter IV** contains the contributions of the author of this thesis to the theory of possibly infinite iterated function systems comprising three main topics, namely: i) the study of the shift space of a possibly infinite iterated function system; ii) some connections between the attractor of a possibly infinite iterated function system $\mathcal{S}$ and the attractors of the sub-IFSs of $\mathcal{S}$; iii) alternative characterizations of hyperbolic affine possibly infinite iterated function systems.

In *Section IV.2* we recall basic definitions and well-known facts concerning Hausdorff-Pompeiu semidistance, possibly infinite iterated function systems (for short IIFS), shift space associated to an IIFS, comparison functions and $\varphi$-contractions. We also introduce the notion of affine possibly infinite iterated function system (AIIFS for short), define its attractor and we mention what does it mean that such a system is hyperbolic, $\varphi$-hyperbolic, point-fibered, uniformly point-fibered, strictly topologically contractive, topologically contractive.

In *Section IV.3* we present a generalization of the notion of the shift space associated to an IFS. More precisely, we describe the relation between this space and the attractor of the IIFS. We construct a canonical projection (which turns out to be continuous) from the shift space of an IIFS on its attractor and provide sufficient conditions for this function to be onto.

*Section IV.4* is devoted to the presentation of some connections between the attractor of an IIFS $\mathcal{S}$ and the attractors of the sub-IFSs of $\mathcal{S}$. More precisely, we present a sufficient condition on a family $(I_j)_{j\in L}$ of nonempty subsets of $I$, where $\mathcal{S} = (X,(f_i)_{i\in I})$ is an IIFS, in order to have the equality $\overline{\bigcup_{j\in L} A_{I_j}} = A$, where $A$ means the attractor of the $\mathcal{S}$ and $A_{I_j}$ means the attractor of the sub-IFS $\mathcal{S}_{I_j} = (X,(f_i)_{i\in I_j})$ of $\mathcal{S}$. In addition, we prove that given an arbitrary infinite cardinal number $\mathcal{A}$ and a complete metric space $(X,d)$, if the attractor of an IIFS $\mathcal{S} = (X,(f_i)_{i\in I})$ is of type $\mathcal{A}$ (this means that there exists a dense subset of it having the cardinal less or equal to $\mathcal{A}$), then there exists $\mathcal{S}_J = (X,(f_i)_{i\in J})$, a sub-IFS of $\mathcal{S}$, having the property that $card(J) \leq \mathcal{A}$, such that the attractors of $\mathcal{S}$ and $\mathcal{S}_J$ coincide.

In *Section IV.5* we prove that if $\mathcal{S} = ((X,\|.\|),(f_i)_{i\in I})$ is an affine possibly infinite iterated function system, then the following statements are equivalent: 1. $\mathcal{S}$ is hyperbolic; 2. there exists a comparison function $\varphi_0$ such that $\mathcal{S}$ is $\varphi_0$-hyperbolic; 3. $\mathcal{S}$ has an attractor; 4. $\mathcal{S}$ is strictly topologically contractive; 5. $\mathcal{S}$ is uniformly point-fibered. It generalized a result due to R. Atkins, M. Barnsley, A. Vince and D.C. Wilson where the case $(X, \|.\|) = (\mathbb{R}^m, \|.\|)$ and $I$ finite is considered.

S.L. Lipscomb introduced the space $L(A)$ and J. C. Perry and Lipscomb proved that the Lipscomb's space $L(A)$ on an arbitrary index set $A$ can be imbedded in Hilbert's space $l^2(A)$, showing that $L(A)$ can be endowed with the metric inherited from $l^2(A)$. Perry showed that $L(A)$ can be injected into the Tychnoff's cube $I^A$ and the imbedded version of $L(A)$ is a subset of a standard $|A|$-simplex $\Delta^A$ which is both a subspace of $l^2(A)$ and a subset of Tychnoff's cube $I^A$. Let $\omega^A$ be the imbedded version of $L(A)$ endowed with the $l^2(A)$-induced topology and let $\omega_c^A$ denote the space whose underlying set is that of $\omega^A$ but whose topology is induced from the Tychnoff's cube $I^A$. Perry showed that $\omega_c^A$ is the attractor of a possibly infinite iterated function system containing affine transformations of $I^A$. He mentions that it is an open problem to construct $\omega^A$ as the attractor of an iterated function system containing an infinite number of affine transformations of $l^2(A)$.

**Chapter V** answers the above mentioned question.

In *Section V.2* we present the Lipscomb space $L(A)$.

In *Section V.3* we show how to construct $\omega^A$ as the attractor of a possibly infinite iterated function system containing affine transformations of $l^2(A)$. In this way we answered the open question of Perry.

In *Section V.4*, by using the results concerning the shift space for an infinite IFS presented in Section IV.3, we show that, for an infinite set $A$, the embedded version of $L(A)$ in $l^p(A)$, $p \in [1, \infty)$, with the metric induced from $l^p(A)$, denoted by $\omega_p^A$, is the attractor of a possibly infinite iterated function system comprising affine transformations of $l^p(A)$. In this way we provide a generalization of the positive answer that we gave to the open problem of Perry.

In *Section V.5* we point out that $\omega_p^A = \omega_q^A$ for all $p, q \in [1, \infty)$ and, by providing a complete description of the convergent sequences from $\omega_p^A$, we prove that the topological structure of $\omega_p^A$ is independent of $p$.

**Chapter VI**, the last one, presents some future plans regarding our professional and scientific career. It describes the following:

a) *the research directions that I intend to follow*, namely:

As short term research projects:

i) the study of new aspects of the connections between the attractor of a possibly infinite iterated function systems, sub-attractors of such a system and type $A$ sets;

ii) the study of a question of A. Kameyama on the existence of a self-similar metric on a topological self-similar set;

iii) the search for a sufficient condition for a finite family of continuous functions on a metric space to be transformed into $\varphi$-contractions.

As long term research project:

iv) invariant vector measures.

b) *the topics that I would like to teach*, attracting in this way the students to the study of iterated function systems.

c) *a new version of the book entitled "Lipschitz functions"* which appeared in 2002 at the Romanian Academy Publishing House.

As a final consideration about the structure of this thesis, I should mention that most results are presented without proofs, but, in order to have a deeper inside, I presented also some sketches of proofs (indicated by `Proof)` which can illustrate better some ideas.